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Boundary algebras of hyperbolic monopoles

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Abstract

We prove the conjecture that a monopole in three-dimensional anti-de Sitter space can be completely determined by its "holographic" image on the conformal boundary two-sphere.
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1. Introduction

Viewed from the conformal two-sphere at infinity there is a fundamental difference between hyperbolic space, which is anti-de Sitter space with positive definite metric, and flat Euclidean space. The isometries of hyperbolic space, $\mathbb{H}^3$, act as the full three complex-dimensional set of conformal transformations on the conformal two-sphere at infinity, S^2_∞, while the isometries of Euclidean space, $\mathbb{R}^3$, act with a large isotropy subgroup on S^2_∞, with only a three real-dimensional set of conformal transformations surviving. In particular, one may detect the location of points in hyperbolic space from observations on S^2_∞. More precisely, a point of $\mathbb{H}^3$ uniquely determines a normal vector field on S^2_∞ by extending geodesics from the point out to infinity. By interpreting the normal vector fields as differential 2-forms on S^2_∞, one can deal with a collection of points in $\mathbb{H}^3$, showing that it uniquely determines the sum of its 2-forms. In contrast, all points in Euclidean space appear the same from S^2_∞, and only the number of points in a collection, not the locations, can be detected.

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A magnetic monopole is considered to be an approximation to a collection of points in space, given by non-linear solitons concentrated at finitely many points. It has a limit at infinity, that appears as a differential 2-form defined on the conformal two-sphere at infinity. From the discussion above, it should not be surprising that in Euclidean space all monopoles with the same charge—essentially the number of points of concentration—look the same on the sphere at infinity, whereas in hyperbolic space a monopole is uniquely determined by its limit at infinity. The latter fact is proven in this paper. Previously, Austin and Braam [5] proved that for a half-integer mass (defined below) SU(2) monopole in $\mathbb{H}^3$ the limit of the monopole on the conformal boundary two-sphere completely determines the monopole, and conjectured it to be true more generally. In this paper we prove the conjecture for any positive real mass SU(2) monopole in $\mathbb{H}^3$.

The concept of field theories being represented by observations at the boundary of space–time has gained much recent interest. In particular, the AdS–CFT correspondence proposes a relationship between string theory on anti-de Sitter space–time and conformal field theory on the boundary [13,19], and this generalises to produce invariants of conformally compact Einstein manifolds with conformal boundary [7]. Both gravity and gauge theory require anti-de Sitter space rather than flat space when gaining information from the conformal sphere at infinity. Further analogues between gravity and gauge theory are suggested by various aspects of this paper such as the tantalising similarity between the mass m of a monopole and the size N of the matrix theory related to gravity. Also, the calculation of n-point functions using solutions of the scattering equation along geodesics in $\mathbb{H}^3$ is analogous to an approximation to the calculation of correlation functions using path integrals appearing in the AdS/CFT correspondence, since the stationary phase approximation reduces the computation of the propagator to the study of the wave equation along geodesics in $\mathbb{H}^3$. (For a closer analogy, perhaps it would be necessary to integrate the n-point functions defined in this paper over the moduli space of monopoles.) We will not comment further on these things in this paper.

The main tool in this paper is an n-point function $\langle P_{z_1} \cdots P_{z_n} \rangle$ defined for a given monopole and any ordered collection of points on the conformal boundary two-sphere $\{z_1, \dots, z_n\} \subset S^2_\infty$. Associated to the ordered collection of points is the set of geodesics in $\mathbb{H}^3$ running from z_1 to z_2 and from z_2 to z_3 and so on until z_n to z_1. The n-point function is a complex number assigned to the sequence of geodesics continuously differentiable in its variables $(z_1, \dots, z_n)$ obtained by solving a scattering equation along the geodesics. The notation $\langle P_{z_1} \cdots P_{z_n} \rangle$ anticipates the construction of an algebra with expectation values given by the n-point function.

The 2-point function is used to settle the open conjecture that the holographic image of the monopole on the conformal boundary two-sphere determines the monopole on hyperbolic space. One can construct an abstract algebra freely generated by the points of S^2_∞ satisfying relations that use the n-point functions. The 3-point function is used to prove that the generators behave like projections, and the 4-point function encodes the fact that the algebra possesses a finite-dimensional representation.

The two proofs that the holographic image of the monopole on the conformal boundary two-sphere determines the monopole on hyperbolic space, given, respectively, for half-integer mass by Austin and Braam [5] and for any mass using the 2-point function, have no a priori relation. The main role of the algebra assembled out of the n-point functions is to supply a relationship between the two methods. A representation of the algebra into a finite-dimensional vector space gives rise to a holomorphic map from the two-sphere to lines in the vector space. Such a holomorphic map arises in the proof by Austin and Braam.

1.1. Main results

Before describing the main results, we will define the objects of the paper. Atiyah [1,2], first studied monopoles over hyperbolic space $\mathbb{H}^3$. A monopole (A, Φ) is a solution of the non-linear Bogomolny equation $d_A\Phi = *F_A$ where A, is a connection defined on a trivial rank two SU(2) bundle E over $\mathbb{H}^3$ with L^2 curvature F_A and the Higgs field $\Phi : \mathbb{H}^3 \to$ SU(2) satisfies $\lim_{r\to\infty}\|\Phi\| = m$, the mass of the monopole. The charge of the monopole is defined to be the topological degree of the map $\Phi_\infty : S^2_\infty \to S^2_\infty$. The hyperbolic metric, featured in the Hodge star $*$, may be replaced by the flat Euclidean metric, giving rise to monopoles in Euclidean space. The gauge group of maps $g : \mathbb{H}^3 \to$ SU(2) acts on the equations and we identify gauge equivalent monopoles. The construction of an n-point function from a monopole is a gauge invariant procedure. On the conformal boundary two-sphere, a monopole has a well-defined limit, given by a $U(1)$ connection [11,18], which we call the holographic image of the monopole.

There is an integrable structure underlying hyperbolic monopoles, best seen on the complex surface of geodesics, $\mathbb{CP}^1 \times \mathbb{CP}^1 - \bar{\Delta}$ (where $\bar{\Delta} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is the anti-diagonal). In the Euclidean case, over its surface of geodesics $T\mathbb{CP}^1$, twistor space techniques are used in [8,9] to understand the construction of monopoles, and the conserved quantities of monopoles. The main tool is the scattering equation:

$$(\partial_t^A - \mathrm{i}\Phi)s = 0 \tag{1}$$

defined for local sections s of E along a geodesic in $\mathbb{R}^3$ parameterised by t. In particular, those geodesics along which an L^2 solution of (1) exists, form a compact algebraic curve inside $T\mathbb{CP}^1$, called the *spectral curve*. Analogously, solutions of (1) along geodesics in $\mathbb{H}^3$ are used to study hyperbolic monopoles [1,2,16] and define the spectral curve of the monopole $\Sigma \subset \mathbb{CP}^1 \times \mathbb{CP}^1 - \bar{\Delta}$.

For $z_1 \neq z_2$, define the 2-point function $\langle P_{z_1} P_{z_2}\rangle$ to be a positive real number associated to (A, Φ) and the geodesic in $\mathbb{H}^3$ joining z_1 and z_2 on the conformal boundary two-sphere as follows. Along this geodesic, choose a solution $s_+(t)$ of (1) that decays as $t \to \infty$. Notice that the parameter t involves a choice of orientation of the geodesic. Choose a decaying solution r_+ of (1) along the same geodesic oriented in the opposite direction. In terms of the oppositely oriented parameter t used in (1), $r_+(t)$ is a solution of the equation:

$$(\partial_t^A + \mathrm{i}\Phi)r = 0 \tag{2}$$

and $r_+(t)$ decays as $t \to -\infty$. The inner product $(r(t), s(t))$ of any two solutions of (1) and (2) is independent of t. If we normalise $r_+(t)$ and $s_+(t)$ by

$$\lim_{t\to\infty} \exp(mt)\|s_+\| = 1, \qquad \lim_{t\to-\infty} \exp(-mt)\|r_+\| = 1, \tag{3}$$

then the decaying solutions are well-defined up to phase and the number $|(r_+, s_+)|^2$ depends only on the geodesic and (A, Φ). Define

$$\langle P_{z_1} P_{z_2} \rangle = |(r_+, s_+)|^2$$

for r_+, s_+ defined along the geodesic joining z_1 and z_2. The n-point function is a complex number defined similarly using decaying solutions of (1) along the set of geodesics running between consecutive points of an ordered n-tuple of points in S^2_∞. For the definition of the n-point function and justification of parts of the definition of the 2-point function given here see Section 2.

Theorem 1.1. *The* 2*-point function uniquely determines the spectral curve of* (A, Φ).

This theorem is rather straightforward. Its power comes from combining it with the deeper theorem that the 2-point function also encodes the holographic image of the monopole on the conformal boundary two-sphere, given by a $U(1)$ connection. The $U(1)$ connection is expressed with respect to a family of gauges related to the spectral curve of the monopole. More explicitly, for each point $w \in S^2_\infty$, the 2-point function enables one to express the $U(1)$ connection over the conformal boundary two-sphere with respect to a gauge defined over the complement of the points $\{z_1, \dots, z_k\}$ that satisfy $(w, z_i) \in \Sigma$, the spectral curve of the monopole. Each such gauge is determined uniquely by properties described in Proposition 2.11. The situation is rigid enough that the $U(1)$ connection uniquely determines the 2-point function.

Theorem 1.2. *The* 2*-point function determines and is determined by the holographic image of the monopole on the conformal boundary two-sphere.*

The spectral curve determines the monopole over hyperbolic space up to gauge equivalence. This is a rather deep non-constructive property of monopoles. It uses the (non-constructive) existence of a trivialisation of a holomorphic line bundle over the spectral curve and sheaf cohomological constructions to retrieve the monopole. Using this we are able to conclude the following corollary.

Corollary 1.3. *The holographic image of the monopole on the conformal boundary two-sphere determines the monopole up to gauge equivalence.*

One might expect Corollary 1.3 to follow from a maximum principle applied to a (non-linear) boundary value problem. This approach was pursued in [17] with only partial success.

An associative algebra can be studied via the values of a linear function, which we call expectation values, defined over the algebra. In some cases, the structure coefficients of the algebra, with respect to a generating set, can be retrieved from the expectation values, thus uniquely determining the algebra. Conversely, one may begin with an abstract set of generators with no a priori algebra structure and use expectation values to define the structure coefficients of the algebra.

Consider the algebra freely generated by the points of the conformal boundary two-sphere, where we notate the generators by P_z, $z \in S^2_\infty$, and add the relations:

$$\exists c = c(z_1, z_2, \dots, z_n) \in \mathbb{C}, \qquad P_{z_1} P_{z_2} \cdots P_{z_n} = c P_{z_1} P_{z_n} \quad \text{when } \langle P_{z_1} P_{z_n} \rangle \neq 0. \tag{4}$$

We suppose that the n-point function defined by a monopole gives the expectation value of the product $P_{z_1} P_{z_2} \cdots P_{z_n}$ and we extend this function linearly to the algebra. Then by taking the expectation values of each side of (4) we can calculate the scalar c. This essentially defines the algebra structure.

The boundary algebra of a monopole is a slight modification of the construction of the previous paragraph. We will add further relations to the algebra in the form of "non-degeneracy" conditions, and enlarge the algebra using derivations.

Definition 1.4. Define the boundary algebra

$$\mathcal{S}(A, \Phi) = \{\mathcal{A}, *, P_z \in \mathcal{A}, z \in S^2_\infty, \langle \cdots \rangle\}$$

for any hyperbolic monopole (A, Φ), to consist of:

(1) an involutive algebra $(\mathcal{A}, *)$ defined over $\mathbb{C}$;
(2) generators $P_z = P_z^*$, for all $z \in S^2_\infty$;
(3) derivations $[\partial_z, \cdot] : \mathcal{A} \to \mathcal{A}$ and $[\partial_{\bar{z}}, \cdot] : \mathcal{A} \to \mathcal{A}$;
(4) further generators $[\partial_z, P_z]$, $[\partial_{\bar{z}}, P_z]$, $[\partial_z, [\partial_z, P_z]]$, $\dots$;
(5) a linear function $\langle \cdots \rangle : \mathcal{A} \to \mathbb{C}$ that restricts to the n-point function of (A, Φ) on products $P_{z_1} P_{z_2} \cdots P_{z_n}$, satisfying $\langle a^* \rangle = \overline{\langle a \rangle}$, $\partial_z \langle a \rangle = \langle [\partial_z, a] \rangle$

with the relations:

(6) $\langle P_{z_1} P_{z_2} \rangle = 0 \Rightarrow P_{z_1} P_{z_2} = 0$;
(7) $\langle a P_z \rangle = 0$ for almost all $z \in S^2_\infty \Rightarrow a = 0$;
(8) $\exists c = c(z_1, z_2, a, b) \in \mathbb{C}$, $P_{z_1} a P_{z_2} = c P_{z_1} b P_{z_2}$ when $P_{z_1} b P_{z_2} \neq 0$.

where $a, b \in \mathcal{A}$.

To give an indication of the various features of the algebra we will mention five properties proven in the paper:

- one can make sense of the 1-point function as the constant function $\langle P_z \rangle \equiv 1$;
- the 2-point function takes its values on the unit interval;
- $P_z^2 = P_z$;
- $P_{z_1} \neq P_{z_2}$ for $z_1 \neq z_2$;
- $P_z[\partial_z, P_z] = 0$.

Identities involving the 4-point function arise when trying to find a representation of the algebra in which the expectation values of observables are given by traces. We have been unable to directly prove these identities, described in the conclusion. Instead we use the fact that such a representation produces a holomorphic map $S^2_\infty \to \mathbb{CP}^k$, where k is the charge of (A, Φ), and compare the setup to something more familiar.

Theorem 1.5. *There exists a finite-dimensional representation of $\mathcal{S}(A, \Phi)$ in which the expectation values are given by traces.*

The holomorphic sphere $S^2_\infty \to \mathbb{CP}^k$ associated to the finite-dimensional representation is reminiscent of that arising in the work of Austin and Braam [5], and proves to be the source of many further interesting properties. It can be obtained without the algebra and gives an alternative proof that the connection on the conformal boundary two-sphere determines the monopole up to gauge equivalence. It also uncovers further features. Amongst these is an application of geometric invariant theory to define the centre of a hyperbolic monopole. One also gets new information regarding rational maps associated to monopoles. Specifically, given a point at infinity, there is a one-to-one mapping between gauge equivalence classes of monopoles and degree k based rational maps $S^2_\infty \to S^2$ well-defined up to a $U(1)$ action. It has never been understood how the rational maps for different points at infinity are related. The holomorphic sphere gives such a relation. These results will appear elsewhere [14].

One can take finite-dimensional sub-algebras of $\mathcal{S}(A, \Phi)$ and find further structure. In the conclusion we describe a family of sub-algebras parameterised by the spectral curve of the monopole. This is particularly interesting due to the conjecture of Atiyah and Murray [3,4] that spectral curves of hyperbolic monopoles may parameterise solutions of the Yang–Baxter equation.

2. The *n*-point function

In this section we will define the n-point function associated to a monopole. As mentioned in Section 1 the geodesics pass near to approximate locations of the monopole and produce an n-point function continuously differentiable in its variables $(z_1, \dots, z_n)$. We will prove that as a geodesic moves out to infinity and away from the monopole, it feels little effect, and thus the limit of the n-point function as two consecutive points come together is the $(n-1)$-point function, and the 1-point function is naturally given by the constant function 1.

2.1. Definition of $\langle \cdots \rangle$

The function $\langle \cdots \rangle$ defined on n-tuples of points in S^2_∞ is invariant under cyclic permutations of the points (and hence behaves like a trace on the boundary algebra). In what follows, we first define the n-point function $\langle P_{z_1} \cdots P_{z_n} \rangle$ for $z_i \neq z_{i+1}$, $z_n \neq z_1$. This is a fundamental quantity in that all other values of $\langle \cdots \rangle$ are derived from it. We use limits to remove the restriction on the n-tuples $\{z_1, \dots, z_n\}$.

Along any geodesic of $\mathbb{H}^3$ parametrised by t, the scattering equations:

$$(\partial_t^A - \mathrm{i}\Phi)s = 0, \qquad (\partial_t^A + \mathrm{i}\Phi)r = 0 \tag{5}$$

are defined for local sections s, r of E. Any pair of solutions has the property that the inner product $(r(t), s(t))$ is independent of t, since

$$\partial_t(r(t), s(t)) = ((\partial_t^A + \mathrm{i}\Phi)r(t), s(t)) + (r(t), (\partial_t^A - \mathrm{i}\Phi)s(t)) = 0.$$

It can be shown [9,12] that that there are solutions s and r unique up to respective constants that decay like $O(\exp(-mt))$ as $t \to \infty$, respectively, like $O(\exp(mt))$ as $t \to -\infty$. Thus two non-trivial solutions s_+, r_+ are uniquely determined up to phase by the conditions that

$$\lim_{t\to\infty} \exp(2mt)\|s_+\|^2 = 1, \qquad \lim_{t\to-\infty} \exp(-2mt)\|r_+\|^2 = 1, \tag{6}$$

$$\langle P_{z_1} \cdots P_{z_n} \rangle, \qquad z_i \neq z_{i+1}, \quad z_n \neq z_1.$$

For distinct $\{z_1, \ldots, z_n\}$, $\langle P_{z_1} \cdots P_{z_n} \rangle$ is a complex number associated to (A, Φ) and the n oriented geodesics in $\mathbb{H}^3$ travelling from z_1 to z_2, then z_2 to z_3 and so on, until z_n to z_1. Notate by r_{12}, s_{12} the solutions r_+, s_+ of (5) along the geodesic running from z_1 to z_2 and r_{23}, s_{23} the solutions r_+, s_+ along the geodesic running from z_2 to z_3 and so on up to r_{n1}, s_{n1}. Further, align the phases of each $r_{i,i+1}, s_{i-1,i}$ as follows. The consecutive solutions s_{12} and r_{23} have the property that they define a common subspace of the fibre of E at z_2 at infinity, or in other words that

$$\lim_{t\to\infty} \exp(mt)s_{12} = c \lim_{t'\to-\infty} \exp(-mt')r_{23}$$

for $c \in \mathbb{C}^*$. Choose r_{23} so that $c = 1$. Similarly, choose a phase for $r_{i,i+1}$ using $s_{i-1,i}$ and for r_{12} using s_{n1}. Define

$$\langle P_{z_1} \cdots P_{z_n} \rangle = (r_{12}, s_{12})(r_{23}, s_{23}) \cdots (r_{n1}, s_{n1}), \tag{7}$$

which depends only on (A, Φ) and the oriented geodesics running in order through $z_1, z_2, \ldots, z_n, z_1$. The 2-point function defined in Section 1 can be obtained by setting $n = 2$ in this construction. In this case the function is real-valued and independent of the orientation of the geodesic and the choice of phases.

Lemma 2.1. *When $z_i \neq z_{i+1}$, $z_n \neq z_1$, the n-point function $\langle P_{z_1} \cdots P_{z_n} \rangle$ is continuously differentiable in $z_1, \ldots, z_n$.*

Proof. Fix $z_2, z_3, \ldots, z_n$ and vary $z_1 = z$. The product on the right hand side of (7) defining $\langle P_z P_{z_2} \cdots P_{z_n} \rangle$ contains the z dependent sections $r_{12}(z)$, $s_{12}(z)$, $r_{n1}(z)$ and $s_{n1}(z)$ with the others constant as z varies. In [9] (and [12] for hyperbolic monopoles) it was shown using a bijection between nearby solutions that the assignment of $r_{12}(z)$, etc., is continuously differentiable in z. Thus, the same is true of inner products involving the z dependent sections, such as $\langle P_z P_{z_2} \cdots P_{z_n} \rangle$. □

For a general n-tuple of points $\{z_1, \ldots, z_n\}$, we define $\langle P_{z_1} \cdots P_{z_n} \rangle$ by continuity. Lemma 2.1 shows that such a definition is consistent. The following lemma explicitly calculates the limits that arise when two points z_i and z_{i+1} come together.

Lemma 2.2. *The 2-point function satisfies $\lim_{z_1\to z_2} \langle P_{z_1} P_{z_2} \rangle = 1$ and the n-point function satisfies $\lim_{z_1\to z_2} \langle P_{z_1} P_{z_2} P_{z_3} \cdots P_{z_n} \rangle = \langle P_{z_2} P_{z_3} \cdots P_{z_n} \rangle$.*

Proof. We will prove only $\lim_{z_1\to z_2}\langle P_{z_1}P_{z_2}\rangle = 1$ since the proof of the limit of the n-point function is essentially the same. We define $\langle P_{z_1}P_{z_2}\rangle = |(r_+, s_+)|^2$ for solutions of (5) satisfying (6). If the connection is trivial and the Higgs field is constant:

$$\partial_t^A \pm \mathrm{i}\Phi = \partial_t \pm \mathrm{i}\begin{pmatrix} \mathrm{i}m & 0 \\ 0 & -\mathrm{i}m \end{pmatrix}, \tag{8}$$

then $r_+ = \exp(mt)(\,1 \quad 0\,)$ and $s_+ = \exp(-mt)(\,1 \quad 0\,)$ so $(r_+, s_+) = 1$ as required.

As $z_1 \to z_2$, the connection and Higgs field become more trivial and constant, respectively. More precisely, there exists a gauge in which

$$\partial_t^A \pm \mathrm{i}\Phi = \partial_t \pm \mathrm{i}\begin{pmatrix} \mathrm{i}m & 0 \\ 0 & -\mathrm{i}m \end{pmatrix} + \epsilon\cdot C\exp(-m|t|), \tag{9}$$

where C is constant and $\epsilon \to 0$ as $z_1 \to z_2$. This follows from Rade [18].

Levinson's theorem [6] uses a contraction mapping argument to show that solutions r_+ on $(-\infty, 0]$ and s_+ on $[0,\infty)$ of (9) (using $\mathrm{i}\Phi$ and $-\mathrm{i}\Phi$, respectively) are in one-to-one correspondence with solutions of (8). Moreover, the norm of the difference between corresponding solutions is controlled by the L^1 norm of the perturbation term $\epsilon\cdot C\exp(-m|t|)$.

In other words, as $z_1 \to z_2$, the solutions r_+ and s_+ tend uniformly to the solutions of (8) on $(-\infty, 0]$ and $[0,\infty)$, respectively, and in fact on any $(-\infty, R]$ and $[-R,\infty)$. The inner product (r_+, s_+) can be calculated at any point $t \in \mathbb{R}$, in particular $t \in [-R, R]$ so $(r_+, s_+) \to 1$ uniformly. □

Thus, we define

$$\langle P_{z_2}^2\rangle := 1, \tag{10}$$

$$\langle P_{z_2}^2 P_{z_3}\cdots P_{z_n}\rangle := \langle P_{z_2}P_{z_3}\cdots P_{z_n}\rangle. \tag{11}$$

Applying the relation (7) given in Definition 1.4 to (11), we get the relation:

$$P_z^2 = P_z, \qquad z \in S^2_\infty, \tag{12}$$

so (10) and (12) allow us to extend the definition of the n-point function to the 1-point function:

$$\langle P_z\rangle := 1 \tag{13}$$

and from this it follows that

$$\langle[\partial_z, P_z]\rangle = 0 = \langle[\partial_{\bar z}, P_z]\rangle. \tag{14}$$

As described in Section 1, expectation values are used to calculate the constant c in relation (8). When $P_{z_1}P_{z_2} = 0$, the expectation values of both sides of (8) are zero, so we instead choose z_0 so that $\langle P_{z_0}P_{z_1}bP_{z_2}\rangle \neq 0$. (By relation (7), z_0 always exists.) Then

$$\langle P_{z_0}P_{z_1}aP_{z_2}\rangle = c(z_1, z_2, a, b)\langle P_{z_0}P_{z_1}bP_{z_2}\rangle \tag{15}$$

enables us to calculate $c(z_1, z_2, a, b)$. This introduces the issue of consistency of the algebra since the constant $c(z_1, z_2, a, b)$ can be calculated in different ways. The following lemma gives the required property of the n-point function.

Lemma 2.3. *For $a \in \mathcal{A}$, $\langle P_{z_0} P_{z_2}\rangle\langle P_{z_0} P_{z_1} P_{z_2} a\rangle = \langle P_{z_0} P_{z_1} P_{z_2}\rangle\langle P_{z_2} a P_{z_0}\rangle$.*

Proof. For $a = P_{z_3} P_{z_4} \cdots P_{z_n}$, where $z_i \neq z_{i+1}$, this follows simply from the definition. Taking limits and derivatives gives the result for general $a \in \mathcal{A}$. □

2.2. Properties

The Bogomolny equation implies that the Higgs field Φ satisfies a maximum principle $\|\Phi\| < m$ where m is the mass of the monopole. This leads to a type of dissipative behaviour of $(\partial_t^A - \mathrm{i}\Phi)$ which can be used to show the following lemma.

Lemma 2.4. *For $z_i \neq z_{i+1}$, $z_n \neq z_1$, $|\langle P_{z_1} \cdots P_{z_n}\rangle| < 1$.*

Proof. Since $\langle P_{z_1} \cdots P_{z_n}\rangle = (r_{12}, s_{12})(r_{23}, s_{23}) \cdots (r_{n1}, s_{n1})$ it is sufficient to show along any geodesic that the solutions s_+, r_+ of (5) satisfy $|(r_+, s_+)| < 1$, and in fact

$$\begin{aligned}|(r_+, s_+)|^2 &= \lim_{t\to-\infty} |(r_+(t), s_+(t))|^2 = \lim_{t\to-\infty} |(\exp(-mt)r_+(t), \exp(mt)s_+(t))|^2\\ &\leq \lim_{t\to-\infty} \|\exp(-mt)r_+(t)\|^2\|\exp(mt)s_+(t)\|^2 = \lim_{t\to-\infty} \|\exp(mt)s_+(t)\|^2\end{aligned}$$

so we will show that $\lim_{t\to-\infty}\|\exp(mt)s_+(t)\|^2 < 1$. We have

$$|\partial_t\|s_+\|^2| = |((\partial_t^A + \mathrm{i}\Phi)s, s) + (s, (\partial_t^A - \mathrm{i}\Phi)s)| = |(2\mathrm{i}\Phi s, s)| < 2m\|s, s\|^2,$$

where the last inequality uses the maximum principle $|\Phi| < m$. Thus

$$\partial_t\|\exp(mt)s_+\|^2 = (2m\|s, s\|^2 + \partial_t\|s_+\|^2)\exp(2mt) > 0.$$

So the function $\|\exp(mt)s_+\|^2$ is strictly increasing, and by construction of s_+, $\lim_{t\to\infty}$ $\|\exp(mt)s_+(t)\|^2 = 1$ yielding the required inequality:

$$\lim_{t\to-\infty} \|\exp(mt)s_+(t)\|^2 < 1.$$ □

Corollary 2.5. *$P_{z_1} \neq P_{z_2}$ for $z_1 \neq z_2$.*

Proof. If $P_{z_1} = P_{z_2}$ then $\langle P_{z_1} P_{z_2}\rangle = \langle P_{z_2}^2\rangle = 1$ which contradicts Lemma 2.4. □

Until now, we have only used the fact that (A, Φ) satisfies the Bogomolny equation very mildly via the maximum principle for Φ and Rade's estimates for the monopole field. Using the full structure of the Bogomolny equation we can show that the assignment $z \mapsto P_z$ possesses a holomorphic property. It is used to prove the most striking properties of the 2-point function and the existence of a useful finite-dimensional representation of $\mathcal{A}$.

With respect to particular local coordinate systems, the Bogomolny equation $\mathrm{d}_A\Phi = *F_A$ decomposes into a holomorphic part and a "moment map" part. Specifically, this occurs for local coordinate systems that reflect the holomorphic structure on the variety of geodesics. Two examples of this are the local coordinates (t, z) in $\mathbb{H}^3$ obtained from a family of geodesics, each parametrised by t, travelling from the fixed $w \in S^2_\infty$ to the varying $z \in S^2_\infty$, and the local coordinates (t, w) in $\mathbb{H}^3$ obtained from a family of geodesics, each parametrised by t, travelling from the varying $w \in S^2_\infty$ to the fixed $z \in S^2_\infty$. The Bogomolny equation decomposes into $[\partial^A_{\bar z}, \partial^A_t - \mathrm{i}\Phi] = 0$, or equivalently, $[\partial^A_z, \partial^A_t + \mathrm{i}\Phi] = 0$, and a second equation which we will omit. Similarly, $[\partial^A_w, \partial^A_t - \mathrm{i}\Phi] = 0$ and the equivalent $[\partial^A_{\bar w}, \partial^A_t + \mathrm{i}\Phi] = 0$ are consequences of the Bogomolny equation. In particular, if r_+ and s_+ are decaying solutions of (5) then

$$\begin{aligned} \partial^A_z r_+ &= \mu_1(w, z) r_+, & \partial^A_{\bar z} s_+ &= \lambda_1(w, z) s_+, \\ \partial^A_{\bar w} r_+ &= \mu_2(w, z) r_+, & \partial^A_w s_+ &= \lambda_2(w, z) s_+ \end{aligned} \tag{16}$$

for (scalar) coefficients μ_i, λ_i *independent* of t. (These are used to obtain a holomorphic bundle, with sub-line bundles on the variety of geodesics of $\mathbb{H}^3$, [2,8].)

Proposition 2.6. *$P_z[\partial_z, P_z] = 0 = [\partial_{\bar z}, P_z]P_z$ and $P_z[\partial_{\bar z}, P_z] = [\partial_{\bar z}, P_z]$.*

Proof. In fact, the three relations are trivially equivalent, so we will prove only $P_z[\partial_{\bar z}, P_z] = [\partial_{\bar z}, P_z]$. Consider the 3-point function:

$$\langle P_{z_1} P_{z_2} P_{z_3} \rangle = \langle P_{z_1} P_{z_2} P_z \rangle = (r_{12}, s_{12})(r_{23}(z), s_{23}(z))(r_{31}(z), s_{31}(z)), \tag{17}$$

where $z_3 = z$ is allowed to vary, z_1 and z_2 are fixed and different from z, and r_{ij}, s_{ij} are the solutions of (5) along the geodesic running from z_i to z_j. We have

$$\partial_{\bar z}\langle P_{z_1} P_{z_2} P_z \rangle = \langle [\partial_{\bar z}, P_{z_1} P_{z_2} P_z] \rangle = \langle P_{z_1} P_{z_2} [\partial_{\bar z}, P_z] \rangle \tag{18}$$

and this will be used to characterise $P_z[\partial_{\bar z}, P_z]$.

By (16) the Bogomolny equation implies that $\partial^A_z r_{23}(z) = \mu(z) r_{23}(z)$ with z dependent coefficient, and $\partial^A_{\bar z} s_{23}(z) = \lambda(z) s_{23}(z)$, since we are moving only one end of the geodesic. The limit $\lim_{t\to-\infty} r_{23}(z)$ is independent of z so we can multiply $r_{23}(z)$ by a function depending on z and arrange that $\mu(z) = 0$, whilst preserving its normalisation at $t = -\infty$. (We cannot do the same for $\lambda(z)$.) Thus

$$\partial_{\bar z}(r_{23}(z), s_{23}(z)) = (\partial^A_z r_{23}(z), s_{23}(z)) + (r_{23}(z), \partial^A_{\bar z} s_{23}(z)) = \lambda(z)(r_{23}(z), s_{23}(z)).$$

If we differentiate (17) then we get

$$\begin{aligned} \langle P_{z_1} P_{z_2} [\partial_{\bar z}, P_z] \rangle &= \lambda(z)(r_{12}, s_{12})(r_{23}(z), s_{23}(z))(r_{31}(z), s_{31}(z)) \\ &\quad + (r_{12}, s_{12})(r_{23}(z), s_{23}(z))\partial_{\bar z}(r_{31}(z), s_{31}(z)). \end{aligned}$$

Let $z_2 \to z$. Then as shown in the proof of Lemma 2.2, $(r_{23}, s_{23}) \to 1$ so

$$\lim_{z_2\to z} \langle P_{z_1} P_{z_2} [\partial_{\bar z}, P_z] \rangle = \lambda(z)(r_{12}, s_{12})(r_{31}, s_{31}) + (r_{12}, s_{12})\partial_{\bar z}(r_{31}, s_{31}) = \langle P_{z_1} [\partial_{\bar z}, P_z] \rangle.$$

Hence $\langle P_{z_1} P_z [\partial_{\bar z}, P_z] \rangle = \langle P_{z_1} [\partial_{\bar z}, P_z] \rangle$ for all z_1, so we get the relation

$$P_z[\partial_{\bar z}, P_z] = [\partial_{\bar z}, P_z]$$

as required. □

We call this a holomorphic relation since it gives a type of integrability condition whereby $\bar{\partial}$ is preserved by P, and since it will translate precisely to an integrability condition when we construct a representation of $\mathcal{A}$.

By Proposition 2.6 $\partial_{\bar{z}}\langle P_w P_z\rangle = \langle P_w P_z[\partial_{\bar{z}}, P_z]\rangle$ so that

$$\langle P_w P_z\rangle = 0 \Rightarrow \partial_{\bar{z}}\langle P_w P_z\rangle = 0.$$

This suggests that it might be fruitful to take some type of log derivative of the 2-point function. In the remainder of this section we will show that the 2-point function, when viewed appropriately, is both a defining function for the spectral curve of the monopole and a Hermitian metric for the connection on the conformal boundary two-sphere.

Lemma 2.7. *The function*

$$\lambda(w, z) = \tfrac{1}{2}\partial_{\bar{z}} \ln \langle P_w P_z\rangle \tag{19}$$

satisfies (i) $\lambda(z, z) = 0$, (ii) $\lambda(w, z)$ *is holomorphic in* w, *and* (iii) $\partial_z\lambda(w, z)$ *is real and independent of* w.

Proof.

(i) We have $2\lambda(z, z) = \lim_{w\to z} \partial_{\bar{z}} \ln \langle P_w P_z\rangle = \lim_{w\to z}\langle P_w[\partial_{\bar{z}}, P_z]\rangle/\langle P_w P_z\rangle$. This can be simplified to $\langle P_z[\partial_{\bar{z}}, P_z]\rangle = \langle[\partial_{\bar{z}}, P_z]\rangle = 0$ by Proposition 2.6 and (14).

(ii) In an open set of $\mathbb{CP}^1 \times \mathbb{CP}^1$ choose solutions of (5) normalised by (6) so that $\lim_{t\to\infty} \exp(mt)s_+(w, z)$ is independent of w and $\lim_{t\to-\infty} \exp(-mt)r_+(w, z)$ is independent of z. (To achieve this choose a normalised solution of (5) $s_+(w_0, z)$ for a fixed $w = w_0$ and use $\lim_{t\to\infty} \exp(mt)s_+(w_0, z) = \lim_{t\to\infty} \exp(mt)s_+(w, z)$ to define $s_+(w, z)$ for nearby w. Do the same for $r_+(w, z)$ around z_0.) Therefore, $\partial^A_w s_+ = 0 = \partial^A_z r_+$ and $\partial^A_{\bar{z}} s_+ = \mu_1(z)s_+$, $\partial^A_{\bar{w}} r_+ = \mu_2(w)r_+$ for $\mu_1(z)$ independent of w and $\mu_2(w)$ independent of z, since we can calculate the coefficients in (16) in the infinite limits. Then (ii) follows from

$$\begin{aligned}\partial_{\bar{w}}\partial_{\bar{z}} \ln \langle P_w P_z\rangle &= \partial_{\bar{w}}\partial_{\bar{z}} \ln |(r_+, s_+)|^2 = \partial_{\bar{w}}\partial_{\bar{z}}(\ln(r_+, s_+) + \ln(s_+, r_+)) \\ &= \partial_{\bar{w}}\mu_1(z) + \partial_{\bar{z}}\mu_2(w) = 0.\end{aligned}$$

(iii) For $\lambda(w, z)$ defined in (19) we can choose a local gauge in which

$$\partial^A_{\bar{z}} s_+(w, z) = \lambda(w, z)s_+(w, z)$$

as follows. Choose $r_+(w, z)$ so that $\partial^A_z r_+(w, z) = 0$ (as in (ii)). Now choose $s_+(w, z)$ so that $(r_+(w, z), s_+(w, z))$ is real. This uniquely determines s_+ up to a constant $U(1)$ gauge transformation given by the ambiguity in the phase of r_+. Then

$$\begin{aligned}\partial_{\bar{z}}\langle P_w P_z\rangle &= \partial_{\bar{z}}|(r_+, s_+)|^2 = \partial_{\bar{z}}(r_+, s_+)^2 = 2(r_+, s_+)(r_+, \partial^A_{\bar{z}} s_+) \\ &= 2\lambda(w, z)(r_+, s_+)^2 = 2\lambda(w, z)\langle P_w P_z\rangle.\end{aligned}$$

For $w' \neq w$ choose the solutions of (5) normalised by (6) along each family of geodesics, respectively, $r'_+(z)$, $s'_+(z)$, $r_+(z)$ and $s_+(z)$, so that $\partial^A_z r_+(z) = 0$ and

$(r_+(z), s_+(z)) \in \mathbb{R}$, and $\partial_{\bar{z}}^A r'_+(z) = 0$ and $\partial_{\bar{z}}^A s'_+(z) = \lambda(w,z)s'_+(z)$ (define s'_+ via $\lim_{t\to\infty}\exp(mt)s'_+(z,t) = \lim_{t\to\infty}\exp(mt)s_+(z,t)$). We can compare $\lambda(w',z)$ and $\lambda(w,z)$ by defining $\theta(z)$ so that $(r'_+(z), \exp(\mathrm{i}\theta(z))s'_+(z)) \in \mathbb{R}$, then

$$\lambda(w',z) = \lambda(w,z) + \mathrm{i}\partial_{\bar{z}}\theta(z).$$

In particular the expression in (iii) is independent of w:

$$\begin{aligned}&\partial_z\partial_{\bar{z}}\ \ln\langle P_{w'}P_z\rangle\,\mathrm{d}z\,d\bar{z}\\&\quad = 2(\partial_z\lambda(w',z) + \partial_{\bar{z}}\bar{\lambda}(w',z))\,\mathrm{d}z\,\mathrm{d}\bar{z}\\&\quad = 2(\partial_z\lambda(w,z) + \partial_{\bar{z}}\bar{\lambda}(w,z) + \mathrm{i}\partial_{\bar{z}}\partial_z\theta - \mathrm{i}\partial_z\partial_{\bar{z}}\theta)\,\mathrm{d}z\,\mathrm{d}\bar{z} = \partial_z\partial_{\bar{z}}\ \ln\langle P_wP_z\rangle\,\mathrm{d}z\,\mathrm{d}\bar{z}\end{aligned}$$

and real since it is the Laplacian of a real-valued function. □

If we replace w in $\lambda(w,z)$ by its antipodal point $\hat{w} = -1/\bar{w}$ then although $\lambda(\hat{w},z)$ is defined only outside the set $\langle P_{\hat{w}}P_z\rangle = 0$, the 2-form

$$\begin{aligned}\tfrac{1}{2}\partial\bar{\partial}\ \ln\langle P_{\hat{w}}P_z\rangle &= \partial_z\lambda(\hat{w},z)\,\mathrm{d}z\,\mathrm{d}\bar{z} + \partial_w\bar{\lambda}(z,\hat{w})\,\mathrm{d}w\,\mathrm{d}\hat{w}\\&\quad + \partial_w\lambda(\hat{w},z)\,\mathrm{d}w\,\mathrm{d}\bar{z} + \partial_z\bar{\lambda}(z,\hat{w})\,\mathrm{d}z\,\mathrm{d}\hat{w}\end{aligned}$$

is well-defined everywhere. To see this, first notice that the term $\partial_w\lambda(\hat{w},z)$ vanishes by Lemma 2.7(ii) and for the same reason $\partial_z\bar{\lambda}(z,\hat{w})\,\mathrm{d}z\,\mathrm{d}\hat{w}$ vanishes. The term $\partial_z\lambda(\hat{w},z)$ is independent of w by Lemma 2.7(iii) so in particular it is well-defined everywhere since for any z we can choose a w such that $\langle P_wP_z\rangle \neq 0$, and the same is true of $\partial_w\bar{\lambda}(z,\hat{w})\,\mathrm{d}w\,\mathrm{d}\hat{w}$. Thus the 2-form $\partial\bar{\partial}\ \ln\langle P_{\hat{w}}P_z\rangle$ is a well-defined closed (1, 1) form. We can use this to prove that the zero set of the real-valued function $\langle P_{\hat{w}}P_z\rangle$ is holomorphic, but instead we will rely on known facts about the spectral curve of the monopole.

Proposition 2.8. *The spectral curve of the monopole is encoded in the* 2*-point function. It is given by*

$$\Sigma = \{(w,z) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | \langle P_{\hat{w}}P_z\rangle = 0\}$$

for $\hat{w}$ the antipodal point of w in $\mathbb{CP}^1$.

Proof. This follows from the simple fact that $\langle P_{\hat{w}}P_z\rangle = 0$ precisely when the solutions r_+, s_+ of (5) decay at both ends, which is the same condition for a geodesic to lie in the spectral curve. Notice that the invariance of Σ under the real structure $(w,z) \mapsto (\hat{z},\hat{w})$ extends to the 2-point function since $\langle P_{\hat{w}}P_z\rangle = \langle P_zP_{\hat{w}}\rangle$. □

We could have equivalently stated Proposition 2.8 in terms of the multiplication operation of the algebra $\mathcal{S}(A,\Phi)$ in place of the 2-point function since $P_{\hat{w}}P_z = 0$ is equivalent to $\langle P_{\hat{w}}P_z\rangle = 0$.

Proposition 2.9. *The connection on the conformal boundary two-sphere is encoded in the* 2*-point function by $\lambda(w,z) = (1/2)\partial_{\bar{z}}\ \ln\langle P_wP_z\rangle$ and*

$$A_\infty = \lambda(w,z)\,\mathrm{d}\bar{z} - \bar{\lambda}(w,z)\,\mathrm{d}z,$$

where w is fixed and gives a choice of gauge. The curvature on the conformal boundary two-sphere is given by $F_{A_\infty} = -\langle [\partial_z, P_z][\partial_{\bar{z}}, P_z]\rangle \,\mathrm{d}z\,\mathrm{d}\bar{z}$.

Proof. Fix w and vary z. The Bogomolny equation implies that the solution s_+ of (5) normalised by (6) also satisfies $\partial^A_{\bar{z}} s_+(z,t) = \lambda(z)s_+(z,t)$ for $\lambda(z)$ independent of t. In the limit, the section $\lim_{t\to\infty}\exp(mt)s_+(z,t)$ gives a unitary gauge for the connection on the conformal boundary two-sphere, and hence $\lambda(z)\,\mathrm{d}\bar{z}$ is the $\mathrm{d}\bar{z}$ component of A_∞. Any other choice of $s_+(z,t)$ satisfying (6) differs by $\exp(\mathrm{i}\theta(z))$ and hence

$$\lambda(z) \mapsto \lambda(z) + \mathrm{i}\partial_{\bar{z}}\theta(z),$$

which is a change of the $U(1)$ gauge. In fact, without the normalisation (6), the $\lambda(z)$ that arises gives the connection on the conformal boundary two-sphere which is Hermitian with respect to a Hermitian metric defined by $\lim_{t\to\infty}\|\exp(mt)s_+(z,t)\|^2$. As in the proof of Lemma 2.7, we can choose $r_+(z)$ and $s_+(z)$ so that $\partial^A_{\bar{z}} r_+(z) = 0$ and $(r_+(z), s_+(z))$ is real. Then $\partial^A_{\bar{z}} s_+ = \lambda(w,z)s_+$ so $\lambda(w,z)\,\mathrm{d}\bar{z}$ gives the $(0,1)$ part of A_∞ with respect to a well-defined $U(1)$ gauge (up to a constant gauge transformation) determined by the choice of w. Thus the first part of the proposition is proven.

The curvature is given by

$$F_{A_\infty} = (\partial_z\lambda(w,z) + \partial_{\bar{z}}\bar{\lambda}(w,z))\,\mathrm{d}z\,\mathrm{d}\bar{z} = \partial_z\partial_{\bar{z}}\ \ln\langle P_w P_z\rangle\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

since $\partial_z\lambda(w,z)$ is real-valued, and

$$\begin{aligned}\partial_z\partial_{\bar{z}}\ \ln\langle P_w P_z\rangle &= \frac{\partial_z\partial_{\bar{z}}\langle P_w P_z\rangle}{\langle P_w P_z\rangle} - \frac{\partial_{\bar{z}}\langle P_w P_z\rangle\partial_z\langle P_w P_z\rangle}{\langle P_w P_z\rangle^2}\\ &= \frac{\langle P_w[\partial_z,[\partial_{\bar{z}},P_z]]\rangle}{\langle P_w P_z\rangle} - \frac{\langle P_w[\partial_{\bar{z}},P_z]\rangle\langle P_w[\partial_z,P_z]\rangle}{\langle P_w P_z\rangle^2}.\end{aligned}$$

This is independent of w, since it is a gauge invariant 2-form or we see it explicitly in Lemma 2.7. Thus we can take the limit $w \to z$ and since $P_z[\partial_z, P_z] = 0$ the second term disappears to leave

$$\partial_z\partial_{\bar{z}}\ \ln\langle P_w P_z\rangle = \langle P_z[\partial_z,[\partial_{\bar{z}},P_z]]\rangle.$$

Since $0 = \langle[\partial_{\bar{z}}, P_z]\rangle = \langle P_z[\partial_{\bar{z}}, P_z]\rangle$ then

$$0 = \partial_z\langle P_z[\partial_{\bar{z}},P_z]\rangle = \langle[\partial_z,P_z][\partial_{\bar{z}},P_z]\rangle + \langle P_z[\partial_z,[\partial_{\bar{z}},P_z]]\rangle$$

thus

$$F_{A_\infty} = \partial_z\partial_{\bar{z}}\ \ln\langle P_w P_z\rangle\,\mathrm{d}z\,\mathrm{d}\bar{z} = -\langle[\partial_z,P_z][\partial_{\bar{z}},P_z]\rangle\,\mathrm{d}z\,\mathrm{d}\bar{z}. \qquad \square$$

The construction of the gauge in which $A_\infty = \lambda(w,z)\,\mathrm{d}\bar{z} - \bar{\lambda}(w,z)\,\mathrm{d}z$ breaks down if $\langle P_w P_z\rangle = 0$. In that case, once $r_+(z)$ is chosen, there is not a unique choice of $s_+(z)$ that satisfies $(r_+(z), s_+(z))$ is real. This simply says that the $U(1)$ gauge defined by w is well-defined, up to locally constant gauge transformations, on the complement of the finite set of points $\{z_1,\ldots,z_k\}$ determined by $\langle P_w P_{z_i}\rangle = 0$, or in other words, w defines a flat structure on a line bundle over $S^2 - \{z_1,\ldots,z_k\}$.

An understanding of the behaviour of A_∞ with respect to the gauge in Proposition 2.9 near its singularities is a key ingredient in the proof that the connection on the conformal boundary two-sphere uniquely determines the 2-point function. Equivalently, we must understand the behaviour of the 2-point function near its zero set.

Lemma 2.10. *Near a point* (w_0, z_0) *in the zero set,* $\langle P_{\widehat{w_0}} P_{z_0}\rangle = 0$*, the function* $\langle P_{\hat{w}} P_z\rangle$ *vanishes like* $|\psi(w,z)|^2$*, where* $\psi(w,z)$ *is a local holomorphic defining function for the zero set.*

Proof. In order to study the vanishing at $\langle P_{\hat{w}_0} P_{z_0}\rangle$, we may ignore the normalisation condition (6) of solutions $r_+(\hat{w}, z)$ and $s_+(\hat{w}, z)$ of (5) since that simply involves multiplying the solutions by non-vanishing functions. Thus we may choose the solutions so that $\partial_z^A r_+ = 0 = \partial_{\bar{z}}^A s_+$ and $\partial_{\bar{w}}^A r_+ = 0 = \partial_w^A s_+$. The inner product $(r_+(\hat{w}, z), s_+(\hat{w}, z))$ is generically a transverse local section of the line bundle $\mathcal{O}(k,k)$ so $|(r_+(\hat{w}, z), s_+(\hat{w}, z))|^2$ vanishes like $|\psi(w,z)|^2$ and so too does $\langle P_{\hat{w}} P_z\rangle$. $\square$

We will summarise the properties of the gauge for A_∞ in the following proposition.

Proposition 2.11. *The* $(0, 1)$ *part of* A_∞*, given by* $\eta_w(z) = \lambda(w,z)\,\mathrm{d}\bar{z}$*, satisfies the properties*:

(1) $\eta_w(z)$ *is well-defined outside a set of points* $\{z_1, \ldots, z_k\}$.
(2) $\eta_w(z) \sim \ln|z - z_i|^2\,\mathrm{d}\bar{z}$ *at each* z_i.
(3) $\mathrm{d}\eta_w(z)$ *is an imaginary valued* 2*-form.*
(4) $\eta_w|_{z=z_0}$ *is holomorphic in* w.
(5) $\eta_w(w) = 0$.

Furthermore, this $U(1)$ *gauge is the unique gauge* (*up to a constant gauge transformation*) *satisfying properties* (1)–(3).

Proof. The points $\{z_1, \ldots, z_k\}$ are determined by $\langle P_w P_{z_i}\rangle = 0$ and Lemma 2.10 determines the behaviour of the singularities there. Properties (3)–(5) follow from Lemma 2.7. Any other 1-form with these properties must differ from $\eta(z)$ by $\mathrm{i}\partial_{\bar{z}}\theta(z)\,\mathrm{d}\bar{z}$ for a real-valued function $\theta(z)$. By (1), $\theta(z)$ is a function defined outside the set of points $\{z_1, \ldots, z_k\}$ and by (2) and (3) it is bounded and harmonic and hence constant. Thus $\mathrm{i}\partial_{\bar{z}}\theta(z)\,\mathrm{d}\bar{z} = 0$ and the properties uniquely determine η. $\square$

Properties (4) and (5) are automatically satisfied by any $\eta(z)$ satisfying (1)–(3). This suggests that the connection on the conformal boundary two-sphere in some sense feels the spectral curve. The next proposition will prove that the connection on the conformal boundary two-sphere does determine the 2-point function and hence the spectral curve.

Proposition 2.12. *The connection on the conformal boundary two-sphere uniquely determines the* 2*-point function.*

Proof. Suppose we have two monopoles (A, Φ) and (A', Φ') with respective algebras consisting of elements P_z and P'_z. Fix w and vary z. The two monopoles have the same connection on the conformal boundary two-sphere precisely when

$$\ln\langle P_w P_z\rangle - \ln\langle P'_w P'_z\rangle \tag{20}$$

is harmonic in $z, \bar{z}$, since the curvatures of the connections on the conformal boundary two-sphere must coincide.

With respect to a local trivialisation of $\mathcal{O}(k, k)$ in the neighbourhood of a point on $\bar{\Delta}$ denote by $\Psi(w, z)$ a section with zero set the spectral curve of (A, Φ), and similarly $\Psi'(w, z)$ for (A', Φ'). Then

$$\ln\langle P_{\hat{w}} P_z\rangle - \ln\langle P'_{\hat{w}} P'_z\rangle + \ln\frac{|\Psi'(w, z)|^2}{|\Psi(w, z)|^2} = \ln\frac{|\Psi'(w, \hat{w})|^2}{|\Psi(w, \hat{w})|^2} \tag{21}$$

since the left hand side of (21) is well-defined everywhere, i.e. we have cancelled singularities, and for fixed w it is harmonic in $z, \bar{z}$. Hence it is constant in z and when we evaluate at $z = \hat{w}$ we get the right hand side.

Now fix z and take $\partial_w\partial_{\bar{w}}$ of both sides of (21). The left hand side vanishes since (20) is also harmonic in $w, \bar{w}$ by symmetry. Thus $\ln|\Psi(w, \hat{w})|^2 - \ln|\Psi'(w, \hat{w})|^2$ is harmonic in $w, \bar{w}$. If $\xi(w)$ is harmonic then it is the sum of a holomorphic and anti-holomorphic function since $\xi + \mathrm{i}\rho$ is holomorphic for some (locally defined $\rho(w)$) and $\xi - \mathrm{i}\rho$ is anti-holomorphic. We can choose Ψ to be real and positive on $\bar{\Delta}$ so $\ln|\Psi(w, \hat{w})|^2 = 2\ln\Psi(w, \hat{w})$ and similarly for Ψ'. Thus

$$\Psi(w, \hat{w}) = g_1(w)g_2(\hat{w})\Psi'(w, \hat{w})$$

for $g_1(w)$ holomorphic and $g_2(\hat{w})$ anti-holomorphic. We conclude that

$$\Psi(w, z) = g_1(w)g_2(z)\Psi'(w, z)$$

since the real analytic function $\Psi(w, \hat{w})$ on $\bar{\Delta}$ has a unique extension in a neighbourhood of $\bar{\Delta} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. But then g_1 and g_2 are both constant since $\Psi_{|\bar{\Delta}} \neq 0$ so the zero set of Ψ cannot contain lines $w = w_0$ or $z = z_0$.

Thus, $\langle P_w P_z\rangle - \langle P'_w P'_z\rangle$ is constant and hence 0 since they agree on $w = z$. □

Remark. This completes the proof of Theorem 1.2 and Corollary 1.3. On closer observation, one soon realises that one of the key facts in the proof of Proposition 2.12—$\Psi(w, \hat{w})$, defined up to multiplication by the norm squared of a holomorphic function, uniquely determines $\Psi(w, z)$ up to a constant—leads to another proof of Corollary 1.3. This viewpoint is taken in [14].

3. Representation

Consider a representation of $\mathcal{S}(A, \Phi)$ on a Hilbert space H that satisfies

$$\langle a\rangle = \operatorname{tr} a \text{ and } a^* \text{ is the adjoint of } a, \tag{22}$$

where we abuse notation and denote $a \in \mathcal{A}$ to also mean its image in the space of endomorphisms of H. The properties $P_z^2 = P_z = P_z^*$ and $\operatorname{tr} P_z = \langle P_z \rangle = 1$ imply that P_z is a *projection* with one-dimensional image. The image of each projection is a line in H so each P_z corresponds to a point in $\mathbb{P}H$ and we have a map $q : S^2_\infty \to \mathbb{P}H$ defined by $q(z) = \operatorname{im} P(z)$. In this section we will describe the properties of $\mathcal{A}$ in terms of the map q. We will defer the proof of existence of a representation until the end of the section. Let k be the charge of the monopole.

Proposition 3.1. *A representation of $\mathcal{S}(A, \Phi)$ on a Hilbert space H satisfying* (22) *gives rise to a* 1–1 *degree k holomorphic map $q : S^2_\infty \to \mathbb{CP}^k$.*

Proof. We will use $|q(z)\rangle$ to label a unit vector in the line $q(z) = \operatorname{im} P(z) \subset H$, and $\langle q(z)|$ its conjugate transpose, so $\langle q(z)|q(z)\rangle = 1$. Thus $|q(z)\rangle$ is still ambiguous up to a phase, although

$$|q(z)\rangle\langle q(z)| = P_z$$

is well-defined.

To show that $q(z)$ is smooth at z_0, choose a w so that $P_w P_{z_0} \neq 0$ and choose a neighbourhood U of z_0 so that $P_w P_z \neq 0$ for $z \in U$. Then fix a unit vector $|q(w)\rangle$ and for each $z \in U$ choose a unit vector $|q(z)\rangle$ so that $\langle q(w)|q(z)\rangle$ is real. Then by Lemma 2.1 $\langle P_w P_z\rangle = \operatorname{tr} P_w P_z = \langle q(w)|q(z)\rangle^2$ is smooth in z so $\langle q(w)|q(z)\rangle$ is smooth in z. Thus the component $P_w q(z)$ of $q(z)$ is smooth. This is true for almost all w so $q(z)$ is smooth on the linear span of the image of q. We may replace H by this linear span, since the representation annihilates the complement. Thus $q(z)$ is a smooth map.

The holomorphicity of $q(z)$ is equivalent to the property $P_z[\partial_{\bar z}, P_z] = [\partial_{\bar z}, P_z]$ proven in Proposition 2.6. This can be seen by setting $P_z = |q(z)\rangle\langle q(z)|$. Then

$$\begin{aligned}&|q(z)\rangle\langle q(z)|(|\partial_{\bar z} q(z)\rangle\langle q(z)| + |q(z)\rangle\langle \partial_z q(z)|)\\ &\quad = (|\partial_{\bar z} q(z)\rangle\langle q(z)| + |q(z)\rangle\langle \partial_z q(z)|) \Rightarrow |q(z)\rangle\langle q(z)|\partial_{\bar z} q(z)\rangle\langle q(z)| = |\partial_{\bar z} q(z)\rangle\langle q(z)|\end{aligned}$$

and by acting on the left by any vector orthogonal to $|q(z)\rangle$ we see that

$$\partial_{\bar z}|q(z)\rangle = \lambda(z)|q(z)\rangle$$

for some function $\lambda(z)$, so $q(z)$ is holomorphic. (We use $\partial_{\bar z}|q(z)\rangle$ and $|\partial_{\bar z} q(z)\rangle$ to mean the same thing.)

The degree of $q(z)$ is obtained by intersecting its image with a hyperplane. This corresponds to asking for the number of solutions z to $P_w P_z = 0$ for a generic w, which is k, the charge of the monopole. Furthermore, the degree of $q(z)$ determines an upper bound for the dimension of the span of its image, thus $q : S^2_\infty \to \mathbb{CP}^k \subset \mathbb{P}H$. The map $q(z)$ is one-to-one since the proof of Corollary 2.5 shows not only that $P_w \neq P_z$ in $\mathcal{A}$ but also that their images under the representation are unequal via $\operatorname{tr} P_w P_z < 1$. □

Proposition 3.2. *The spectral curve of a charge k* SU(2) *hyperbolic monopole with associated holomorphic sphere $q : S^2_\infty \to \mathbb{CP}^k$ is given by*

$$\Sigma = \{(w, z) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | (q(\hat w), q(z)) = 0\}$$

where $\hat{w}$ *is the antipodal point of* w *and* $(\cdot,\cdot)$ *the natural Hermitian product on* $\mathbb{C}^{k+1}$. *Equivalently,* $w^k(q(\hat{w}), q(z)) = \psi(w,z)$, *the defining polynomial of* Σ.

Proof. This is simply a restatement of Proposition 2.8 since the product of two projections is zero precisely when their images are orthogonal. The function $(q(\hat{w}), q(z))$ is quite different from the corresponding function $\langle P_{\hat{w}} P_z\rangle$. In particular it is holomorphic, and hence can be represented by a polynomial. $\square$

Recall from [5] that to an SU(2) integral mass charge k hyperbolic monopole one can associate a solution of the discrete Nahm equations. In the following $m \in \mathbb{Z} + 1/2$:

$$\begin{aligned}
&\gamma_j = \gamma_{-j}^{\mathrm{T}}, \quad -2m+2 \le j \le 2m-2, \quad j \text{ odd},\\
&\beta_j = \beta_{-j}^{\mathrm{T}}, \quad -2m+1 \le j \le 2m-1, \quad j \text{ even},\\
&\beta_{j-1}\gamma_j - \gamma_j\beta_{j+1} = 0, \quad -2m+2 \le j \le 2m-2, \quad j \text{ odd},\\
&[\beta_j^*, \beta_j] + \gamma_{j-1}^*\gamma_{j-1} - \gamma_{j+1}\gamma_{j+1}^* = 0, \quad -2m+3 \le j \le 2m-3, \quad j \text{ even},\\
&[\beta_{2m-1}^*, \beta_{2m-1}] + v\bar{v}^{\mathrm{T}} - \gamma_{2m-2}^*\gamma_{2m-2} = 0,
\end{aligned}$$

where $\beta_i, \gamma_j \in \mathbf{gl}(k,\mathbb{C})$ and $v \in \mathbb{C}^k$ admit an action of $\{g_j \in U(k) | j = -2m+1, -2m+3, \ldots, 0, \ldots, 2m-3, 2m-1, g_j = \bar{g}_{-j}\}$ by

$$\beta_j \mapsto g_j\beta_j g_j^{-1}, \qquad \gamma_j \mapsto g_{j-1}\gamma_j g_{j+1}^{-1}, \qquad v \mapsto g_{-2m+1}v.$$

(Note that we have replaced v with v^{T} from [5] so that the vector v is a column vector and matrices act on its left.) The pair (β_{-2m+1}, v) determines the full solution of the discrete Nahm equations. It was shown in [5] that the map

$$\begin{pmatrix} \beta_{-2m+1} - z \\ v^{\mathrm{T}} \end{pmatrix} : \mathbb{C}^k \to \mathbb{C}^{k+1} \tag{23}$$

is a monad on S^2 which determines the boundary value of the hyperbolic monopole. The monad can be interpreted as a degree k holomorphic map $\beta : S^2 \to \mathbb{CP}^k$ given explicitly by

$$\beta(z) = \begin{pmatrix} -\det(\beta_{-2m+1} - z)\cdot(\beta_{-2m+1}^{\mathrm{T}} - z)^{-1}v, \\ \det(\beta_{-2m+1} - z). \end{pmatrix}. \tag{24}$$

The map is well-defined up to the $U(k)$ action on the first k coordinates, since β_{-2m+1} admits a $U(k)$ action. The map β has the properties that the pull-back of the Kähler form $\beta^*\omega$ gives the curvature of the monopole on the conformal boundary two-sphere (and hence its gauge equivalence class). Furthermore, by a theorem of Calabi the pull-back of the Kähler form, and hence the curvature of the monopole on the conformal boundary two-sphere, uniquely determines the map β. Thus the boundary value of the monopole determines the monopole.

Proposition 3.3. *The spectral curve of* (A, Φ) *is given by*

$$\Sigma = \{(w,z) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | (\beta(\hat{w}), \beta(z)) = 0\}.$$

Proof. This is a simple result from linear algebra. For any two vectors $u, v \in \mathbb{C}^n$:

$$\det(1 + u\bar{v}^{\mathrm{T}}) = 1 + (v, u) \tag{25}$$

since $(u, v) \mapsto (g^{-1}u, \bar{g}^{\mathrm{T}}v)$ preserves (25) for any $g \in \mathrm{GL}(n, \mathbb{C})$, so we may assume $u = (1, 0, 0, \dots)$, in which case (25) is easy.

Put $\mathrm{d}(w, z) = \det(\bar{\beta}_{-2m+1} + 1/w)\det(\beta_{-2m+1} - z)$ for ease in reading the next set of formulae:

$$\begin{aligned}(\beta(\hat{w}), \beta(z)) &= \mathrm{d}(w, z)\left(\bar{v}^{\mathrm{T}}\left(\bar{\beta}_{-2m+1} + \frac{1}{w}\right)^{-1}(\beta^{\mathrm{T}}_{-2m+1} - z)^{-1}v + 1\right)\\ &= \mathrm{d}(w, z)\det\left(1 + (\beta^{\mathrm{T}}_{-2m+1} - z)^{-1}v\bar{v}^{\mathrm{T}}\left(\bar{\beta}_{-2m+1} + \frac{1}{w}\right)^{-1}\right) \text{ by (25)}\\ &= \det\left((\beta_{-2m+1} - z)\left(\bar{\beta}_{-2m+1} + \frac{1}{w}\right) + v\bar{v}^{\mathrm{T}}\right)\end{aligned}$$

and the last expression defines the spectral curve by specialising the expression in [15] to the boundary value of the discrete Nahm equations. □

Corollary 3.4. *For half-integer mass, the holomorphic map $q : S^2_\infty \to \mathbb{CP}^k$ associated to the algebra $\mathcal{S}(A, \Phi)$ coincides up to the action of $U(k+1)$ on its image with the holomorphic map $\beta : S^2 \to \mathbb{CP}^k$ arising from the discrete Nahm equations.*

Strictly, we should say that in the $U(k + 1)$ orbit of the map $q : S^2_\infty \to \mathbb{CP}^k$ associated to the algebra $\mathcal{S}$, there is a $U(k)$ orbit of the map β.

Proof. The expressions

$$w^k(\beta(\hat{w}), \beta(z)) \quad \text{and} \quad w^k(q(\hat{w}), q(z))$$

coincide since they both define holomorphic sections of $\mathcal{O}(k, k)$ with the same zero set. Thus $\beta(z) = uq(z)$ for some $u \in U(k + 1)$. □

Remark. Another corollary of Proposition 3.3 is a new proof of the fact that the boundary value of the monopole determines the monopole when the mass is a half-integer.

Proposition 3.5. *There exists a representation of $\mathcal{S}(A, \Phi)$ on a Hilbert space H that satisfies $\langle a\rangle = \mathrm{tr}\, a$ and a^* is the adjoint of a for $a \in \mathcal{A}$.*

Proof. In [14] it is proven that for each charge k monopole (A, Φ) there exists a holomorphic map $q : S^2_\infty \to \mathbb{CP}^k$ with two key properties. It determines and is determined by the spectral curve of (A, Φ) and satisfies the statement of Proposition 3.2, and it determines and is determined by the boundary value A_∞ of (A, Φ). The curvature of A_∞ is obtained as the pull-back of the Kähler form on $\mathbb{CP}^k$ by q.

As in the proof of Proposition 3.1, use $|q(z)\rangle$ to label a unit vector in the line $q(z)$, and $\langle q(z)|$ its conjugate transpose, so $|q(z)\rangle\langle q(z)| = R_z$ is well-defined. We will prove that $R_z = R_z^*$ is the image of P_z in a representation of $\mathcal{A}$ acting on $\mathbb{C}^{k+1}$ satisfying $\langle P_{z_1} \cdots P_{z_n} \rangle = \mathrm{tr}\, R_{z_1} \cdots R_{z_n} = \langle q(z_1)|q(z_2)\rangle\langle q(z_2)|q(z_3)\rangle \cdots \langle q(z_n)|q(z_1)\rangle$. Since $\langle a \rangle$ for any $a \in \mathcal{A}$ is obtained from derivatives and limits of such quantities, this is enough to show the representation satisfies (22).

The functions $\langle P_w P_z \rangle$ and $|\langle q(w)|q(z)\rangle|^2$ vanish to the same order on (an image under $w \mapsto \hat{w}$ of) the spectral curve of (A, Φ) and vanish nowhere else. Thus

$$\langle P_w P_z \rangle = \xi(w, z)|\langle q(w)|q(z)\rangle|^2$$

for a real-valued nowhere vanishing function $\xi(w, z)$. Fix $q(w)$ and choose $q(z)$ so that $\langle q(w)|q(z)\rangle \in \mathbb{R}$ for each z. Take the derivative of each side with respect to $\partial_{\bar{z}}$ so

$$2\lambda(w, z)\langle P_w P_z \rangle = (2\lambda(w, z) + \partial_{\bar{z}} \ln \xi(w, z))\xi(w, z)|\langle q(w)|q(z)\rangle|^2$$

since both $\langle P_w P_z \rangle$ and $|\langle q(w)|q(z)\rangle|^2$ define $A_\infty = \lambda(z)\,\mathrm{d}\bar{z} - \bar{\lambda}(z)\,\mathrm{d}z$. Hence

$$\partial_{\bar{z}} \ln \xi(w, z) = 0$$

so $\xi(w, z)$ is constant. It is identically 1 since $\langle P_z^2 \rangle = 1 = |\langle q(z)|q(z)\rangle|^2$.

Note that our assumption that $\langle P_w P_z \rangle$ and $|\langle q(w)|q(z)\rangle|^2$ define the same gauge for A_∞ is unnecessary since if they differ by the gauge transformation:

$$\lambda(w, z) \mapsto \lambda(w, z) + \mathrm{i}\partial_{\bar{z}}\theta(w, z)$$

for a real-valued $\theta(w, z)$, then we are left with $\partial_{\bar{z}} \ln \xi(w, z) = -2\mathrm{i}\partial_{\bar{z}}\theta(w, z)$ in which case ξ is harmonic and hence constant, thus $\theta \equiv 0$. The general case is proved analogously. Again since we know the vanishing behaviour of the respective functions, we have

$$\langle P_{z_1} \cdots P_{z_n} \rangle = \xi(z_1, \ldots, z_n)\langle q(z_1)|q(z_2)\rangle\langle q(z_2)|q(z_3)\rangle \cdots \langle q(z_n)|q(z_1)\rangle$$

for a nowhere vanishing ξ. Vary z_1 and fix the other variables. Choose $q(z_1)$ so that $\langle q(z_1)|q(z_2)\rangle \in \mathbb{R}$ for each z_1. Then again

$$2\lambda(z_2, z_1)\langle P_{z_1} \cdots P_{z_n} \rangle = (2\lambda(z_2, z_1) + (\partial_{\bar{z}_1} \ln \xi))\langle P_{z_1} \cdots P_{z_n} \rangle$$

and $\partial_{\bar{z}_1} \ln \xi(z_1, \ldots, z_n) = 0$. Thus ξ is constant and it is 1 on the diagonal $z_i = z_1$, so it is identically 1. □

Proposition 2.9 shows that the charge at infinity is

$$F_{A_\infty} = -\langle [\partial_z, P_z][\partial_{\bar{z}}, P_z] \rangle\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

and we expect that it takes on only one sign, i.e. $F_{A_\infty}/2\pi\mathrm{i}$ to be non-negative with respect to the orientation $\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}$ since it is true for half-integer mass monopoles. This is a consequence of the following corollary of Proposition 3.5 which uses the positivity of the trace on the product of a matrix with its adjoint.

Corollary 3.6. *$\langle a^* a \rangle \geq 0$ for any $a \in \mathcal{A}$, with equality precisely when $a = 0$.*

Furthermore, using knowledge of when $\langle a^* a\rangle$ is zero, we can understand the singularities of q in terms of the curvature. Since $\partial_{\bar z}|q(z)\rangle = \lambda(z)|q(z)\rangle$, q is singular at z_0 if and only if $\partial_z|q(z)\rangle_{|z_0} = \mu|q(z_0)\rangle$ for some $\mu \in \mathbb{C}$. Now

$$0 = \partial_z\langle q(z)|q(z)\rangle_{|z_0} = \langle \partial_{\bar z} q(z)|q(z)\rangle_{|z_0} + \langle q(z)|\partial_z q(z)\rangle_{|z_0} = \bar\lambda(z_0) + \mu$$

thus $[\partial_z, P_z]_{|z_0} = \partial_z|q(z)\rangle\langle q(z)|_{|z_0} = (\bar\lambda(z_0) + \mu)|q(z_0)\rangle\langle q(z_0)| = 0$. So by Corollary 3.6, q has a singularity at z_0 if and only if $F_{A_\infty}(z_0) = 0$.

4. Conclusion

The important features of $\mathcal{S}(A, \Phi)$ have thus far used the bounded, real-valued 2-point function $\langle P_w P_z\rangle$. The 3-point function was needed to prove some of the properties of $\langle P_w P_z\rangle$. Since the 2-point function determines the algebra it might be that one need not look much further to the n-point functions. On the other hand, there are features of $\mathcal{S}(A, \Phi)$ that have yet to be understood and may require the higher order functions:

(i) The existence of a finite-dimensional representation of $\mathcal{S}(A, \Phi)$ with expectation values of observables given by the trace implies relations amongst the 4-point functions. More precisely, for a charge k monopole, choose a generic set of points $\{z_i | i = 0, \ldots, N\}$ (where N is the dimension of the span of the image of $q(z)$, so $N = k$ if q is "full") and set $P_i = P_{z_i}$. Then the finite-dimensional representation allows any P_w to be expressed as $\alpha^{ij}(w)P_i P_j$ (sum repeated indices) where the $\alpha^{ij}(w)$ are determined via $\langle P_w P_k P_l\rangle = \alpha^{ij}(w)\langle P_i P_j P_k P_l\rangle$. Set $g_{ijkl} = \langle P_i P_j P_k P_l\rangle$. Then (for generic choice $\{z_i | i = 0, \ldots, N\}$) there exists an "inverse" g^{ijkl} satisfying $g^{ijkl} g_{klmn} = \delta_{im}\delta_{jn}$, so $\alpha^{ij}(w) = g^{ijkl}\langle P_w P_k P_l\rangle$. Then

$$\langle P_w P_z\rangle = g^{ijkl}\langle P_w P_k P_l\rangle\langle P_z P_i P_j\rangle.$$

If we multiply both sides by the "determinant" of g_{ijkl} then the relation holds for all sets $\{z_i | i = 0, \ldots, N\}$, and not just generic sets. It would be more satisfying to be able to prove the relations directly and use this to get the representation.

(ii) It would be interesting to recognise the mass of the monopole in $\mathcal{S}(A, \Phi)$. The mass is encoded in the spectral curve but it is difficult to extract.

(iii) Since $\mathcal{S}(A, \Phi)$ brings the spectral curve of (A, Φ) and the connection on the conformal boundary two-sphere closer together, one might hope to understand both the metrics of Austin and Braam [5] and Hitchin [10] from a similar perspective.

(iv) One can take finite-dimensional sub-algebras of $\mathcal{S}(A, \Phi)$ to possibly uncover further structure. In the case $k = 2$, define $\mathcal{S}_w(A, \Phi) \subset \mathcal{S}(A, \Phi)$ to be the sub-algebra generated by $P_1(w) = P_{z_1}$ and $P_2(w) = P_{z_2}$ where $P_w P_{z_i} = 0$. This is a finite-dimensional algebra, generated as a vector space by $P_1(w)$, $P_2(w)$, $P_1(w)P_2(w)$ and $P_2(w)P_1(w)$. The algebra $\mathcal{S}_w(A, \Phi)$ actually depends on a point in the spectral curve of the monopole, since the elements $P_1(w)$ and $P_2(w)$ are ordered.

The algebra $\mathcal{S}(A, \Phi)$ of an SU(2) hyperbolic monopole generalises to any gauge group. In such a case, the scattering equations (5) admit solutions with various rates of decay. To each point $z \in S^2_\infty$ we associate finitely many operators, one for each level of decay of solutions of the scattering equation, with given relations. The n-point functions are obtained from pairing solutions of the scattering equations with specified decay in each direction. For higher rank Lie groups, just as the operators P_z define one-dimensional subspaces of a very large vector space to give a holomorphic map $q : S^2_\infty \to \mathbb{CP}^k$, the finitely many operators associated to $z \in S^2_\infty$ will define a flag inside a very large vector space with a corresponding holomorphic map. The dimension of the vector space will be determined by the charge of the monopole, as in Proposition 3.1.

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